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Non-semisimple Lie algebras with Levi factor $\mathfrak{so}(3)$, $\mathfrak{sl}(2,\mathbb{R})$ and their invariants

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Abstract

We analyse the number $\mathcal{N}$ of functionally independent generalized Casimir invariants for non-semisimple Lie algebras $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ with Levi factors isomorphic to $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$ in dependence of the pair $(R,\mathfrak{r})$ formed by a representation R of $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$. We show that for any dimension $n \geqslant 6$, there exist Lie algebras $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ with non-trivial Levi decomposition such that $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 0$.

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1. Introduction

The important role played by invariant theory in physics was recognized long ago. Electroweak interactions and quantum numbers in the study of particle states are based on the concept of symmetry, and their invariants provide fundamental information. Among the various types of symmetry, dynamical ones constitute one of the more important cases, as shown by Gell-Mann and Ne'eman in their hadron classification [1]. The analysis of the group $SU(3)$ resulted in the prediction of new particles whose mass could be derived from the invariants of the group. The invariants of Lie algebras have also shown their effectiveness in the description of Hamiltonians [2], the labelling of irreducible representations or the study of coadjoint orbits [3, 4]. Other important applications of invariants arise in their combination with the theories of Lie algebra contractions, deformations and rigidity [5–8]. For example, all kinematical algebras are related by a contraction procedure, which has allowed a further analysis of these algebras [7, 9, 10]. The interest of invariants of rigid Lie algebras is fully justified by the fact that semisimple Lie algebras are rigid. The invariants of semisimple Lie algebras constitute a classical problem, and it is the only case which has been solved in a satisfactory manner. The invariants of solvable Lie algebras are only studied for specific classes, as they do not underlie a structure theory such as the classical algebras. What refers to the Lie algebras with non-trivial Levi decomposition, invariants are known for physically important algebras, such as the special affine algebras $\mathfrak{sa}(n, \mathbb{R})$, the kinematical Lie algebras and their subalgebras.

A formula for the number $\mathcal{N}(\mathfrak{g})$ of functionally independent invariants of the coadjoint representation of a Lie algebra $\mathfrak{g}$ was given by Beltrametti and Blasi [11] and Pauri and Prosperi [12] in the mid-1960s. This fact reduces the computation of this number to the determination of the rank of a skew-symmetric matrix $A(\mathfrak{g})$ whose entries correspond to the Lie brackets of $\mathfrak{g}$. With some effort, this formula can be used to show that the number of invariants of semisimple Lie algebras coincides with its rank [13]. Moreover, it proves that for direct sums $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of Lie algebras, the number $\mathcal{N}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ is $\mathcal{N}(\mathfrak{g}_1) + \mathcal{N}(\mathfrak{g}_2)$. One can ask whether for semi-direct sums $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$, $\mathfrak{s}$ being the Levi factor of $\mathfrak{g}$, R a representation of $\mathfrak{s}$ and $\mathfrak{r}$ the maximal solvable ideal (called radical) of $\mathfrak{g}$, some formula exists which allows one to express $\mathcal{N}(\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r})$ in terms of $\mathcal{N}(s)$, $\mathcal{N}(\mathfrak{r})$ and some quantity related to the representation R. The motivation of this problem lies in the study of the special affine algebras $\mathfrak{sa}(n, \mathbb{R})$, which are a semidirect sum of the simple Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ and an n-dimensional Abelian Lie algebra [14]. These algebras are known to have only one invariant (which turns out to be a Casimir operator), which shows that the representation plays a crucial role in the semidirect product, and that in principle the existence of a formula expressing the number of invariants in terms of the factors does not exist. The main reason for its nonexistence lies in the distinct possibilities of choice for radicals $\mathfrak{r}$ for a fixed representation of $\mathfrak{s}$. The question that arises naturally in this context is if there exist Lie algebras $\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ with non-trivial Levi decomposition (i.e. $\mathfrak{s} \neq 0$ and $[\mathfrak{s}, \mathfrak{r}] \neq 0$) such that $\mathcal{N}(\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}) = 0$.

In this work, we show that such algebras exist for any dimension $n \geqslant 6$. Moreover, by considering the simple algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$, we analyse the number $\mathcal{N}(\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r})$ for various kinds of representations R and solvable Lie algebras $\mathfrak{r}$.

Any Lie algebra $\mathfrak{g}$ considered in this work is defined over the field $\mathbb{R}$ of real numbers. We convene that nonwritten brackets are either zero or obtained by antisymmetry. We also use the Einstein summation convention. Abelian Lie algebras of dimension n will be denoted by nL_1.

2. Invariants of Lie algebras: the Beltrametti–Blasi formula

The method to determine the invariants of a Lie algebra in terms of systems of partial differential equations (PDEs) has become standard in the physical literature [15, 16], and it is the one we will use here. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ and $\left\{C_{ij}^k\right\}$ be the structure constants over this basis. We consider the representation of $\mathfrak{g}$ in the space $C^\infty(\mathfrak{g}^*)$ given by

$$\widehat{X}_i = -C_{ij}^k x_k \partial_{x_j} \tag{1}$$

where $[X_i, X_j] = C_{ij}^k X_k$ $(1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n)$. This representation is isomorphic to $\mathrm{ad}(\mathfrak{g})$, and therefore satisfies the brackets $[\widehat{X}_i, \widehat{X}_j] = C_{ij}^k \widehat{X}_k$. The invariants $F(X_1, \ldots, X_n)$ of $\mathfrak{g}$

$$[X_i, F(X_1, \ldots, X_n)] = 0 \tag{2}$$

are found by solving the system of linear first-order partial differential equations

$$\widehat{X}_i F(x_1, \ldots, x_n) = -C_{ij}^k x_k \partial_{x_j} F(x_1, \ldots, x_n) = 0 \qquad 1 \leqslant i \leqslant n \tag{3}$$

and then replacing the variables x_i by the corresponding generator X_i (possibly after symmetrizing). In recent years, new algorithms to solve system (3) have been developed, which simplify the calculation in some cases [16]. A maximal set of functionally independent solutions of (3) will be called a fundamental set of invariants. Polynomial solutions of system (3) are therefore polynomials in the generators which commute with $\mathfrak{g}$, thus correspond to the well-known Casimir operators [15]. The system does not impose additional conditions which imply that the solutions are polynomials, so that a non-polynomial solution will be called, in some analogy with the classical case, a generalized Casimir invariant or simply an invariant of $\mathfrak{g}$. If F reduces to a constant we say that the invariant is trivial. In the case of semisimple Lie algebras, the solutions found are in fact Casimir operators, and the number of functionally independent invariants is given by the dimension of its Cartan subalgebra. However, for non-semisimple Lie algebras, there is no reason to suppose that only the polynomial invariants are of physical interest. A classical example for a Hamiltonian being a nonlinear function of the Casimir operators was described by Pauli in [17].

Another important task is to find the maximal number $\mathcal{N}(\mathfrak{g})$ of functionally independent solutions of (3). For the case of the classical groups this number depends only on the dimension of a Cartan subalgebra, while for solvable Lie algebras no such general formula exists [5]. However, for a fixed algebra, this number can be described in terms of the dimension and a certain matrix associated with the commutator table. More specifically, we denote by $A(\mathfrak{g})$ the matrix representing the commutator table of $\mathfrak{g}$ over a given basis, i.e.,

$$A(\mathfrak{g}) = \left(C_{ij}^{k} x_k\right). \tag{4}$$

Such a matrix has necessarily even rank. Then $\mathcal{N}(\mathfrak{g})$ is given by

$$\mathcal{N}(\mathfrak{g}) = \dim \mathfrak{g} - \sup_{x_1,\dots,x_n} \left\{\operatorname{rank}\left(C_{ij}^{k} x_k\right)\right\}. \tag{5}$$

This formula was first described by Beltrametti and Blasi [11] and Pauri and Prosperi [12]. The number of polynomial solutions is generally lower than $\mathcal{N}(\mathfrak{g})$, up to certain special classes of Lie algebras (such as semisimple and nilpotent) [15].

Invariants of Lie algebras have been determined for some classes of non-semisimple Lie algebras, such as solvable Lie algebras in low dimensions [18, 19], the kinematical Lie algebras [7] or the special affine Lie algebras [16].

We give an example to illustrate the general method of obtaining the invariants. Let $\mathfrak{s} = \mathfrak{so}(3)$ and consider the representation $R = \operatorname{ad}\mathfrak{so}(3)$. Let us suppose that the radical of the six-dimensional Lie algebra $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ is the three-dimensional Abelian algebra $3L_1$. The algebra $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ is of interest for multidimensional extensions of the Bianchi type-IX cosmology [20], and the corresponding vacuum Einstein field equations have been solved in [20]. Indeed, this is the simplest embedding of a Bianchi type-IX algebra in an algebra with non-trivial Levi decomposition [21]. It can easily be verified that $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ satisfies $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 2$. The invariants are solution of the system

$$\left.\begin{aligned}
\widehat{X}_1F &= \left(-x_3\partial_{x_2} + x_2\partial_{x_3} - x_6\partial_{x_5} + x_5\partial_{x_6}\right)F = 0\\
\widehat{X}_2F &= \left(x_3\partial_{x_1} - x_1\partial_{x_3} + x_6\partial_{x_4} - x_4\partial_{x_6}\right)F = 0\\
\widehat{X}_3F &= \left(-x_2\partial_{x_1} + x_1\partial_{x_2} - x_5\partial_{x_4} + x_4\partial_{x_5}\right)F = 0\\
\widehat{X}_4F &= \left(-x_6\partial_{x_2} + x_5\partial_{x_3}\right)F = 0\\
\widehat{X}_5F &= \left(x_6\partial_{x_1} - x_4\partial_{x_3}\right)F = 0\\
\widehat{X}_6F &= \left(-x_5\partial_{x_1} + x_4\partial_{x_2}\right)F = 0
\end{aligned}\right\}. \tag{6}$$

Since the equations $\{\widehat{X}_iF = 0\}_{i=4,5,6}$ do not depend on $\partial_{x_i}F$ for $i = 4, 5, 6$, we can extract the following system from (6):

$$\left.\begin{aligned}
\widehat{X}'_1F &= \left(-x_6\partial_{x_5} + x_5\partial_{x_6}\right)F = 0\\
\widehat{X}'_2F &= \left(x_6\partial_{x_4} - x_4\partial_{x_6}\right)F = 0\\
\widehat{X}'_3F &= \left(-x_5\partial_{x_4} + x_4\partial_{x_5}\right)F = 0
\end{aligned}\right\} \tag{7}$$

which has the solution $I_1 = x_4^2 + x_5^2 + x_6^2$. Now, as the rank of the coefficient matrix corresponding to this subsystem is two, the other solution of (6) will depend also on x_1, x_2, x_3. This invariant can be chosen as $I_2 = x_1x_4 + x_2x_5 + x_3x_6$. The important fact about this example is the solution found extracted from the subsystem (7). In the following section, we will see that this is not casual, but a property that holds in general.

3. Semidirect sums of Lie algebras

The classification of Lie algebras is simplified in some manner by the Levi decomposition theorem, which states that any Lie algebra is essentially formed from a semisimple Lie algebra $\mathfrak{s}$ called the Levi factor of $\mathfrak{g}$ and a maximal solvable ideal $\mathfrak{r}$, called the radical [22]. Since the latter is an ideal, the Levi factor $\mathfrak{s}$ acts on $\mathfrak{r}$, and there are two possibilities for this action:

$$[\mathfrak{s}, \mathfrak{r}] = 0 \qquad [\mathfrak{s}, \mathfrak{r}] \neq 0.$$

If the first holds, then $\mathfrak{g}$ is a direct sum $\mathfrak{s} \oplus \mathfrak{r}$, whereas the second possibility implies the existence of a representation R of $\mathfrak{s}$ which describes the action, i.e.,

$$[x, y] = R(x)y \qquad \forall x \in \mathfrak{s} \quad y \in \mathfrak{r}. \tag{8}$$

Unless there is no ambiguity, it is more convenient to write $\overrightarrow{\oplus}_R$ instead of $\overrightarrow{\oplus}$, which is the common symbol for denoting semidirect products. Since (8) implies that the radical is a module over $\mathfrak{s}$, we have to expect severe restrictions on the structure of the radical, while for direct sums any solvable Lie algebra is suitable as radical [23].

Proposition 1. *Let $\mathfrak{s}$ be a semisimple Lie algebra and R an irreducible representation. If $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ is the Levi decomposition of a Lie algebra, then $\mathfrak{r}$ is an Abelian algebra.*

The proof is immediate, since the Jacobi condition implies that the ideals $\mathfrak{r}^{(0)} := \mathfrak{r}$, $\mathfrak{r}^{(i)} := [\mathfrak{r}^{(i-1)}, \mathfrak{r}^{(i-1)}]$ for $i \geqslant 1$ are invariant by the action of $\mathfrak{s}$. If R is irreducible, then either $\mathfrak{r}^{(1)} = 0$ or $\mathfrak{r}^{(1)} = \mathfrak{r}$, and since $\mathfrak{r}$ is solvable, $\mathfrak{r}^{(1)} \neq \mathfrak{r}$. Reasoning similarly, we can easily deduce that the radical $\mathfrak{r}$ is mapped into its maximal nilpotent ideal $\mathfrak{n}$ (usually called the nilradical of $\mathfrak{r}$), from which the following property follows.

Proposition 2. *Let $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ be a Levi decomposition. If the representation R does not possess a copy of the trivial representation, then the radical $\mathfrak{r}$ is a nilpotent Lie algebra.*

This result is in some manner surprising, since it implies the existence of a copy of the trivial representation whenever the radical is not nilpotent. Of course, it does not imply that a nilpotent Lie algebra cannot be the radical when the representation contains copies of the trivial representation.

The Lie algebras having non-trivial Levi decomposition have been completely classified up to dimension nine [23]. For dimension ten, some partial results do also exist, mainly Levi factors isomorphic to rank one simple Lie algebras. Since the algebra $\mathfrak{so}(3)$ is a compact form of $\mathfrak{sl}(2, \mathbb{C})$, the number of (real) representations of $\mathfrak{so}(3)$ is lower than that for $\mathfrak{sl}(2, \mathbb{R})$ [24], which implies the existence of much more Lie algebras having the latter as Levi factor.

Lemma 1. *Let $\mathfrak{g} = \mathfrak{s} \oplus r$. Then $\mathcal{N}(\mathfrak{g}) = \mathcal{N}(\mathfrak{s}) + \mathcal{N}(\mathfrak{r})$.*

This is an obvious consequence of the Beltrametti–Blasi formula. Since the sum is direct, we have that $[\mathfrak{s}, \mathfrak{r}] = 0$ and therefore the rank of the matrix $A(\mathfrak{g})$ is the sum of the ranks of $A(\mathfrak{s})$ and $A(\mathfrak{r})$. Now one can ask what happens whenever we have a non-trivial Levi decomposition. Here no apparent relation between the number of invariants of the Levi factor and the radical, and the number of invariants of the semidirect sum seems to exist. If we consider the simple algebra $\mathfrak{sl}(2,\mathbb{R}) = \{X_1, X_2, X_3 \mid [X_1, X_2] = 2X_2, [X_1, X_3] = -2X_3, [X_2, X_3] = X_1\}$ and the representation $R = D_{\frac{1}{2}} \oplus D_0$, $D_{\frac{1}{2}}$ being the irreducible representation of highest weight $\lambda = 1$, there are two choices of $\mathfrak{r}$ such that $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}\mathfrak{r}$ is a six-dimensional Lie algebra with non-trivial Levi decomposition: either the three-dimensional Heisenberg Lie algebra $\mathfrak{h}_1 = \{X_4, X_5, X_6 \mid [X_4, X_5] = X_6\}$ or the algebra $A_{3,3} = \{X_4, X_5, X_6 \mid [X_i, X_6] = X_i, i = 4, 5\}$ (see [19] for this notation). It is a straightforward verification that $\mathcal{N}(\mathfrak{h}_1) = \mathcal{N}(A_{3,3}) = 1$, thus the formula, if existing, should give the same value for both cases. Now the Lie algebra $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_R\mathfrak{h}_1$ admits two (polynomial) invariants $I_1 = x_6$ and $I_2 = 2x_1x_4x_5 + 4x_2x_3x_6 + 2x_2x_5^2 - 2x_3x_4^2 + x_1^2x_6$, while the algebra $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}A_{3,3}$ has no invariant. The conclusion is that the number of invariants will, in general, not be expressible in terms of its factors. This example points out another interesting fact: the existence of pairs $(R,\mathfrak{r})$ formed by representations R of a semisimple Lie algebra $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$ with structure of $\mathfrak{s}$-module such that

$$\mathcal{N}(\mathfrak{r}) > 0 \qquad \text{and} \qquad \mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 0. \tag{9}$$

This also shows that it is not sufficient to determine the invariants of solvable Lie algebras to have an overview of invariants of Lie algebras, implicitly assumed in some early works. Thus the Levi decomposition theorem does not simplify the determination of Casimir operators of Lie algebras, up to the case where we obtain a direct sum. The next step is naturally to try the classification of pairs $(R,\mathfrak{r})$ for fixed Levi factor $\mathfrak{s}$ such that (9) holds. This problem cannot be solved in a satisfactory manner since the classification of solvable Lie algebras is, in practice, not possible for dimensions $n \geqslant 7$ (the classification of six-dimensional real Lie algebras contains some errors and some omissions). Although an algorithm to obtain all solvable Lie algebras has been proposed in [25], the formidable computations involved in higher dimensions make this classification extremely difficult to be effectively implemented. We must therefore restrict ourselves to certain special cases that are of interest, either for mathematical or physical reasons.

Table 1 shows the complete list of Lie algebras in dimension $\leqslant 8$ with non-trivial Levi decomposition and having no invariants. All of them are indecomposable, up to the seventh algebra, which is a direct sum of the six-dimensional algebra and the two-dimensional affine Lie algebra $\mathfrak{r}_2$. Due to the low dimensions, the only Levi factors that appear are the simple Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$. These algebras are of own interest, since they play an important role in multidimensional cosmologies [26]. The algebras of table 1 have been calculated explicitly imposing the condition that the rank of the matrix $A(\mathfrak{g})$ equals the dimension of $\mathfrak{g}$, although they can also be deduced from the classifications presented in [23, 27]. The advantage of the direct determination used to construct table 1 is that it allows a precise insight into the effect of the representation R of the Levi part on the radical $\mathfrak{r}$, as well as the structure of the elements of $\mathfrak{r}$ which do not belong to its derived subalgebra $[\mathfrak{r},\mathfrak{r}]$.

We convene that the term D_J denotes the real representation of $\mathfrak{sl}(2,\mathbb{R})$ in its standard form, while R_4 denotes the four-dimensional real irreducible representation of $\mathfrak{so}(3)$ and D_0 denotes the trivial representation in both cases.

Although the general classification of these algebras does not seem to be realizable, since it is based on the possibility of classifying the solvable Lie algebras, once an example is known we can deduce the following generic result.

Table 1. Lie algebras with non-trivial Levi factor and $\mathcal{N} = 0$.

Levi factor $\mathfrak{s}$	Dim	Representation	Nonzero structure constants
$\mathfrak{sl}(2, \mathbb{R})$	6	$D_{\frac{1}{2}} \oplus D_0$	$C_{12}^2 = 2, C_{13}^3 = -2, C_{23}^1 = 1, C_{14}^4 = 1, C_{15}^5 = -1,$ $C_{25}^4 = 1, C_{34}^5 = 1, C_{46}^4 = 1, C_{56}^5 = 1$
$\mathfrak{so}(3)$	8	$R_4 \oplus D_0$	$C_{12}^3 = 1, C_{13}^2 = -1, C_{23}^1 = 1, C_{14}^7 = \frac{1}{2}, C_{15}^6 = \frac{1}{2},$ $C_{16}^5 = -\frac{1}{2}, C_{17}^4 = -\frac{1}{2}, C_{24}^5 = \frac{1}{2}, C_{25}^4 = -\frac{1}{2}, C_{26}^7 = \frac{1}{2},$ $C_{27}^6 = -\frac{1}{2}, C_{34}^6 = \frac{1}{2}, C_{35}^7 = -\frac{1}{2}, C_{36}^4 = -\frac{1}{2}, C_{37}^5 = \frac{1}{2},$ $C_{48}^4 = 1, C_{58}^5 = 1, C_{68}^6 = 1, C_{78}^7 = 1$
$\mathfrak{so}(3)$	8	$R_4 \oplus D_0$	$C_{12}^3 = 1, C_{13}^2 = -1, C_{23}^1 = 1, C_{14}^7 = \frac{1}{2}, C_{15}^6 = \frac{1}{2},$ $C_{16}^5 = -\frac{1}{2}, C_{17}^4 = -\frac{1}{2}, C_{24}^5 = \frac{1}{2}, C_{25}^4 = -\frac{1}{2}, C_{26}^7 = \frac{1}{2},$ $C_{27}^6 = -\frac{1}{2}, C_{34}^6 = \frac{1}{2}, C_{35}^7 = -\frac{1}{2}, C_{36}^4 = -\frac{1}{2}, C_{37}^5 = \frac{1}{2},$ $C_{48}^4 = p, C_{48}^6 = -1, C_{58}^5 = p, C_{58}^7 = -1, C_{68}^4 = 1,$ $C_{68}^6 = p, C_{78}^5 = 1, C_{78}^7 = p$
$\mathfrak{sl}(2, \mathbb{R})$	8	$2D_{\frac{1}{2}} \oplus D_0$	$C_{12}^2 = 2, C_{13}^3 = -2, C_{23}^1 = 1, C_{14}^4 = 1, C_{15}^5 = -1,$ $C_{16}^6 = 1, C_{17}^7 = -, C_{25}^4 = 1, C_{27}^6 = 1, C_{34}^5 = 1,$ $C_{36}^7 = 1, C_{48}^4 = 1, C_{58}^5 = 1, C_{68}^4 = 1, C_{68}^6 = 1,$ $C_{78}^5 = 1, C_{78}^7 = 1$
$\mathfrak{sl}(2, \mathbb{R})$	8	$2D_{\frac{1}{2}} \oplus D_0$	$C_{12}^2 = 2, C_{13}^3 = -2, C_{23}^1 = 1, C_{14}^4 = 1, C_{15}^5 = -1,$ $C_{16}^6 = 1, C_{17}^7 = -, C_{25}^4 = 1, C_{27}^6 = 1, C_{34}^5 = 1,$ $C_{36}^7 = 1, C_{48}^4 = 1, C_{58}^5 = 1, C_{68}^6 = p, C_{78}^7 = p\ (p \neq -1)$
$\mathfrak{sl}(2, \mathbb{R})$	8	$2D_{\frac{1}{2}} \oplus D_0$	$C_{12}^2 = 2, C_{13}^3 = -2, C_{23}^1 = 1, C_{14}^4 = 1, C_{15}^5 = -1,$ $C_{16}^6 = 1, C_{17}^7 = -1, C_{25}^4 = 1, C_{27}^6 = 1, C_{34}^5 = 1,$ $C_{36}^7 = 1, C_{48}^4 = p, C_{48}^6 = -1, C_{58}^5 = p, C_{58}^7 = -1,$ $C_{68}^4 = 1, C_{68}^6 = p, C_{78}^5 = 1, C_{78}^7 = p\ (p \neq 0)$
$\mathfrak{sl}(2, \mathbb{R})$	8	$D_{\frac{1}{2}} \oplus 3D_0$	$C_{12}^2 = 2, C_{13}^3 = -2, C_{23}^1 = 1, C_{14}^4 = 1, C_{15}^5 = -1,$ $C_{25}^4 = 1, C_{34}^5 = 1, C_{46}^4 = 1, C_{56}^5 = 1, C_{78}^8 = 1$
$\mathfrak{sl}(2, \mathbb{R})$	8	$D_{\frac{3}{2}} \oplus D_0$	$C_{12}^2 = 2, C_{13}^3 = -2, C_{23}^1 = 1, C_{14}^4 = 3, C_{15}^5 = 1,$ $C_{16}^6 = -1, C_{17}^7 = -3, C_{25}^4 = 3, C_{26}^5 = 2, C_{27}^6 = 1,$ $C_{34}^5 = 1, C_{35}^6 = 2, C_{36}^7 = 3, C_{48}^4 = 1, C_{58}^5 = 1,$ $C_{68}^6 = 1, C_{78}^7 = 1$

Theorem 1. *Let $\mathfrak{s}$ be a semisimple Lie algebra and $(R, \mathfrak{r})$ be a pair formed by a representation of $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$ such that $\mathcal{N}(\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}) = 0$. Then, for any $k \geqslant 1$, there exists a Lie algebra $\mathfrak{g}_k$ with Levi factor $\mathfrak{s}$ and dimension $n = \dim(\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}) + 2k$ such that $\mathcal{N}(\mathfrak{g}) = 0$.*

Proof. Consider the Lie algebra $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_{R'} \mathfrak{r}'$, where $R' = R \oplus 2kD_0$ and the radical is $\mathfrak{r}' = \mathfrak{r} \oplus k\mathfrak{r}_2$, where $\mathfrak{r}_2$ is the affine Lie algebra generated by Y, Z and brackets $[Y, Z] = Z$. The algebra $\mathfrak{r}'$ is obviously a $\mathfrak{s}$-module, and since $\mathcal{N}(\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}) = 0$ and $\mathcal{N}(\mathfrak{r}_2) = 0$, the assertion follows from lemma 1. □

Corollary 1. *For any dimension $n \geqslant 6$ there exist Lie algebras $\mathfrak{g}$ with non-trivial Levi decomposition such that $\mathcal{N}(\mathfrak{g}) = 0$.*

This reduces the classification to the pairs $(R, \mathfrak{r})$ formed by a representation of $\mathfrak{s}$ (this being fixed) and radicals $\mathfrak{r}$ which are indecomposable, i.e., that do not decompose into a direct sum of ideals. Even for low dimensions like ten, it is far from being easy to find such pairs.

As an example, consider the representation $R = R_4 \oplus 3D_0$ of $\mathfrak{so}(3)$ and the radical $\mathfrak{r}$ defined by the brackets

$$\begin{array}{lll} [X_i, X_8] = X_i & 4 \leqslant i \leqslant 7 & \\ [X_4, X_9] = X_6 & [X_5, X_9] = X_7 & [X_6, X_9] = -X_4 \\ [X_7, X_9] = -X_5 & [X_9, X_{10}] = X_{10} & \end{array}$$

over the basis $\{X_4, \ldots, X_{10}\}$. This is the simplest non-decomposable solvable Lie algebra such that the semidirect sum $\mathfrak{so}(3)\overrightarrow{\oplus}_R\mathfrak{r}$ has no non-trivial invariants (for the considered representation). In fact more is true, namely the nonexistence of solvable Lie algebras $\mathfrak{r}$ such that the action of the generators $X \in \mathfrak{r} - [\mathfrak{r}, \mathfrak{r}]$ over the nilradical $[\mathfrak{r}, \mathfrak{r}]$ is diagonal. This will happen also for other representations different from the one taken here.

4. Levi factors $\mathfrak{s} = \mathfrak{so}(3), \mathfrak{sl}(2, \mathbb{R})$

Theorem 1 is a general result which holds for any Lie algebra satisfying (9), and therefore not dependent on the particular Levi factor taken. Now an inspection of table 1 points out some interesting facts for the considered Levi factors $\mathfrak{s} = \mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$. In this section, we analyse the semidirect sums $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ with these Levi subalgebras in more detail. Throughout this section, and unless otherwise stated, the notation $\mathfrak{s}$ will refer either to $\mathfrak{sl}(2, \mathbb{R})$ or to $\mathfrak{so}(3)$.

We saw in section 2 that in the computation of the invariants of the algebra $\mathfrak{so}(3)\overrightarrow{\oplus}_R\mathfrak{r}$ with $R = \mathrm{ad}\,\mathfrak{so}(3)$ and $\mathfrak{r}$ the three-dimensional Abelian algebra $3L_1$, there was an invariant depending only on the variables associated with $3L_1$. We claimed that the existence of this invariant, coming from a special subsystem of (6), was not casual. The next proposition shows that this property does not depend on the representation.

Theorem 2. *Let R be an irreducible representation of $\mathfrak{s}$. Then the semidirect sum $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ admits non-trivial invariants. Moreover, if $\dim(\mathfrak{r}) > \dim(\mathfrak{s})$, there exists a fundamental set of invariants formed by functions F_i depending only on variables associated with elements of $\mathfrak{r}$.*

Proof. We prove it for $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$, the case of $\mathfrak{so}(3)$ being similar. First, we only need to prove the result for odd-dimensional representations D_j, since the remaining case follows at once from the odd dimensionality of the semidirect sum. By proposition 1, the radical $\mathfrak{r}$ is Abelian, and the maximal weight of R is $\lambda = 2m - 4\,(m \geqslant 3)$. Let $\{X_1, X_2, X_3, \ldots, X_{2m}\}$ be a basis of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ such that $\{X_1, X_2, X_3\}$ is a basis of $\mathfrak{sl}(2, \mathbb{R})$ (with $[X_1, X_2] = 2X_2, [X_1, X_3] = -2X_3, [X_2, X_3] = X_1$) and $\{X_4, \ldots, X_{2m}\}$ is a basis of the Abelian radical $\mathfrak{r}$. The system of PDEs giving the invariants of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ is

$$\left.\begin{array}{l} \widehat{X}_1F = \left(-2x_2\partial_{x_2} + 2x_3\partial_{x_3} - \sum_{i=0}^{2m-4}(\lambda - 2i)x_{4+i}\partial_{x_{4+i}}\right)F = 0 \\ \widehat{X}_2F = \left(2x_2\partial_{x_1} - x_1\partial_{x_3} - \sum_{i=1}^{2m-4}(\lambda - i + 1)x_{3+i}\partial_{x_{4+i}}\right)F = 0 \\ \widehat{X}_3F = \left(-2x_3\partial_{x_1} + x_1\partial_{x_2} - \sum_{i=0}^{2m-5}(i + 1)x_{5+i}\partial_{x_{4+i}}\right)F = 0 \\ \widehat{X}_{4+i}F = \left((\lambda - 2i)x_{4+i}\partial_{x_1} - (i + 1)x_{5+i}\partial_{x_2} + (\lambda - i + 1)x_{3+i}\partial_{x_3}\right)F = 0 \\ \qquad\qquad 0 \leqslant i \leqslant 2m - 4 \end{array}\right\}. \tag{10}$$

Observe that since $\mathfrak{r}$ is Abelian, the equations $\{\widehat{X}_{4+i}F = 0\}_{0\leqslant i\leqslant 2m-4}$ do not involve the partial derivatives $\partial_{x_i}F$ for $4 \leqslant i \leqslant 2m$. This allows us to extract the subsystem

$$\left.\begin{array}{l} \widehat{X}'_1F = \left(\sum_{i=0}^{2m-4}(\lambda - 2i)x_{4+i}\partial_{x_{4+i}}\right)F = 0 \\ \widehat{X}'_2F = \left(\sum_{i=1}^{2m-4}(\lambda - i + 1)x_{3+i}\partial_{x_{4+i}}\right)F = 0 \\ \widehat{X}'_3F = \left(\sum_{i=0}^{2m-5}(i + 1)x_{5+i}\partial_{x_{4+i}}\right)F = 0 \end{array}\right\} \tag{11}$$

and any solution is obviously an invariant of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$. The question reduces to show that the system (11) admits a non-trivial solution for any irreducible representation D_J. Observe that (11) can be written as

$$\begin{pmatrix} \lambda x_4 & (\lambda-2)x_5 & \cdots & -(\lambda-2)x_{2m-1} & -\lambda x_{2m} \\ 0 & \lambda x_4 & \cdots & 2x_{2m-2} & x_{2m-1} \\ x_5 & 2x_6 & \cdots & \lambda x_{2m} & 0 \end{pmatrix} \begin{pmatrix} \partial_{x_4}F \\ \cdot \\ \cdot \\ \partial_{x_{2m}}F \end{pmatrix} = 0. \tag{12}$$

Now this matrix of coefficients has at most rank three (indeed three if $m \geqslant 4$ and rank one if $m = 3$), so that (11) has always a solution, which shows that $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) \neq 0$. In particular, the system (11) gives the following number of solutions:

$$\left.\begin{matrix} 1 & \text{if} & m = 3 \\ 2m-6 & \text{if} & m \geqslant 4 \end{matrix}\right\}. \tag{13}$$

Observe that for $m = 3$ the representation R is the adjoint representation, and in this case we can find another invariant which depends also on the variables x_1, x_2, x_3. For $m \geqslant 4$ it is not difficult to see that $\partial_{x_i}F = 0$ for $i = 1, 2, 3$, which shows that the $(2m-6)$ functionally independent solutions of (11) constitute a fundamental set of invariants for $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$. □

Corollary 2. *Let $s = \mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(3)$. If the radical $\mathfrak{r}$ is Abelian then $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) \neq 0$.*

Proof. If the representation contains a copy of the trivial representation D_0 or $\dim\mathfrak{r}$ is even, we automatically have solutions of the corresponding system (3). If R does not contain a copy of D_0, we can again extract a subsystem from (3), since the radical is Abelian and its equations do not contain the partial derivatives corresponding to elements of $\mathfrak{r}$. Now R is a sum of irreducible representations, of which at least one summand R_0 must have even highest weight λ, in order to ensure the odd dimensionality of $\mathfrak{r}$. Moreover, the variables involved in R_0 do not appear in the other summands of $\mathfrak{r}$, which ensures that we can apply the preceding theorem. This shows that there exists a nontrivial of the subsystem corresponding to R_0, which, by the complete reducibility of R and the abelianity of $\mathfrak{r}$, is also an invariant of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$. □

The following example illustrates the procedure used in this proof: let $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R})$ and consider the reducible representation $D_1 \oplus D_{\frac{1}{2}}$. Suppose that the radical $\mathfrak{r}$ is a five-dimensional Abelian Lie algebra. The invariants of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ are the solutions of the system

$$\left.\begin{array}{l} \left(-2x_2\partial_{x_2} + 2x_3\partial_{x_3} - 2x_4\partial_{x_4} + 2x_6\partial_{x_6} - x_7\partial_{x_7} + x_8\partial_{x_8}\right)F = 0 \\ \left(-2x_2\partial_{x_1} + x_1\partial_{x_3} + 2x_4\partial_{x_5} + x_5\partial_{x_6} + x_7\partial_{x_8}\right)F = 0 \\ \left(2x_3\partial_{x_1} - x_1\partial_{x_2} + x_5\partial_{x_4} + 2x_6\partial_{x_5} + x_8\partial_{x_7}\right)F = 0 \\ \left(-2x_4\partial_{x_1} - x_5\partial_{x_3}\right)F = 0 \\ \left(-2x_4\partial_{x_2} - 2x_6\partial_{x_3}\right)F = 0 \\ \left(2x_6\partial_{x_1} - x_5\partial_{x_2}\right)F = 0 \\ \left(-x_7\partial_{x_1} - x_8\partial_{x_3}\right)F = 0 \\ \left(x_8\partial_{x_1} - x_7\partial_{x_2}\right)F = 0 \end{array}\right\}. \tag{14}$$

We extract a subsystem from the first three equations

$$\left.\begin{array}{l} \left(-2x_4\partial_{x_4} + 2x_6\partial_{x_6} - x_7\partial_{x_7} + x_8\partial_{x_8}\right)F = 0 \\ \left(2x_4\partial_{x_5} + x_5\partial_{x_6} + x_7\partial_{x_8}\right)F = 0 \\ \left(x_5\partial_{x_4} + 2x_6\partial_{x_5} + x_8\partial_{x_7}\right)F = 0 \end{array}\right\} \tag{15}$$

Table 2. Ten-dimensional indecomposable Lie algebras with a compact subalgebra of dimension $n \geqslant 7$.

Algebra	Levi decomposition	Representation R	$\mathcal{N}$
$L_{10,14}$	$\mathfrak{so}(3)\overrightarrow{\oplus}_R(7L_1)$	R_7	4
$L_{10,15}$	$\mathfrak{so}(3)\overrightarrow{\oplus}_R(7L_1)$	$R_4 \oplus \mathrm{ad}\,\mathfrak{so}(3)$	4
$L_{10,27}$	$\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_R(7L_1)$	D_3	4
$L_{10,28}$	$\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_R(7L_1)$	$D_2 \oplus D_{\frac{1}{2}}$	4
$L_{10,29}$	$\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_R(7L_1)$	$D_{\frac{3}{2}} \oplus D_1$	4
$L_{10,30}$	$\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_R(7L_1)$	$D_1 \oplus 2D_{\frac{1}{2}}$	4

and any solution of this system is an invariant of the algebra. Equation (15) can also be reduced to

$$\left.\begin{aligned}\left(-2x_4\partial_{x_4}+2x_6\partial_{x_6}\right)F&=0\\ \left(2x_4\partial_{x_5}+x_5\partial_{x_6}\right)F&=0\\ \left(x_5\partial_{x_4}+2x_6\partial_{x_5}\right)F&=0\end{aligned}\right\} \tag{16}$$

which is the subsystem corresponding to the adjoint representation. Clearly, the polynomial $I_1 = 4x_4x_6 - x_5^2$ is a solution of (14) and (15), and therefore an invariant of the algebra. Since the other summand of R is $D_{\frac{1}{2}}$, the other invariant will depend on all the variables $x_4, \ldots, x_8$. We find $I_2 = x_4x_8^2 - x_5x_7x_8 + x_6x_7^2$. Thus I_1, I_2 form a fundamental set of invariants of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$.

These two preceding results constitute an important restriction for a semidirect sum $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ to satisfy $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 0$. Any representation in such algebra must be reducible and contain a copy of the trivial representation D_0 (see table 1 and the examples in section 3).

Proposition 3. *Let $\mathfrak{s} = \mathfrak{so}(3), \mathfrak{sl}(2,\mathbb{R})$. If the radical $\mathfrak{r}$ of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ has a one-dimensional centre, then the representation R describing the semidirect sum contains a copy of the trivial representation D_0. In particular, $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) \neq 0$.*

Proof. Let z generate the centre $Z(\mathfrak{r})$ or $\mathfrak{r}$. For any $X \in \mathfrak{s}$ and $Y \in \mathfrak{r}$ we have

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

which shows that $[\mathfrak{s}, Z(\mathfrak{r})] \subset Z(\mathfrak{r})$. Now

$$[X_2,[X_3,Z]] + [Z,[X_2,X_3]] + [X_3,[Z,X_2]] = 0$$

which shows that $[X_1, Z] = 0$. Similarly it is proven that $[X_2, Z] = [X_3, Z] = 0$, from which we deduce the existence of a copy of the trivial representation in the decomposition of R. Since the action of $\mathfrak{s}$ over $Z(\mathfrak{r})$ is zero, we will obtain the monomial invariant $I_1 = z$. □

The results obtained so far for the Levi factors $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$ have important physical applications, such as the classification of multidimensional spacetimes [21]. In this frame, all ten-dimensional real Lie algebras having a $(7+d)$-dimensional compact subalgebra have been determined. Of special interest are those which have non-trivial Levi decomposition, and which are the only candidates that could present the anomaly $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 0$. From the 30 classes found [26], only 6 are indecomposable, i.e., they do not decompose as a direct sum of lower-dimensional Lie algebras. They have been listed in table 2, where the notation for the algebras is the same as in [26].

By theorem 2 and corollary 2 we see that, since the radical is always Abelian, we will obtain non-trivial invariants. For these algebras, in contrast to the possible multidimensional cosmological models seen in section 3 and table 1, the existence of a compact subalgebra of dimension $n \geqslant 7$ implies that the algebra has non-vanishing invariants.

5. Application to radicals with a codimension one Abelian ideal

In this section, we analyse a special kind of radicals. We will suppose that $\mathfrak{r}$ is a solvable non-nilpotent Lie algebra such that $[\mathfrak{r}, \mathfrak{r}]$ is a codimension one Abelian ideal. We will see that such radicals always imply the existence of invariants, up to the lower-dimensional cases. In particular, the radicals found in table 1 for the eight-dimensional algebras $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ will constitute the exception for radicals of this type.

Theorem 3. *Suppose that $R = R' \oplus 2D_0$, where R' is a representation of $\mathfrak{s}$. Then $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) > 0$.*

Proof. Since R contains at least two copies of the trivial representation, there exists an element $Y \in [\mathfrak{r}, \mathfrak{r}]$ such that $[\mathfrak{s}, Y] = 0$. Let $T \notin [\mathfrak{r}, \mathfrak{r}]$ and $[T, Y] = \sum_{Y_i \in [\mathfrak{r},\mathfrak{r}]} a_i Y_i (a_i \in \mathbb{R})$. The equation $\widehat{Y}F = 0$ of system (3) has the form

$$\widehat{Y}F = -\left(\sum_{Y_i \in [\mathfrak{r},\mathfrak{r}]} a_i y_i\right) \partial_T F = 0. \tag{17}$$

Now, if $[T, Y] = 0$, the function $F = y$ is an invariant of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$. If the bracket $[T, Y]$ is nonzero, then (17) implies that $\partial_T F = 0$ for any invariant F. The complete reducibility of the representation R (the ideal $[\mathfrak{r}, \mathfrak{r}]$ has codimension one in $\mathfrak{r}$ and is an $\mathfrak{s}$-module) implies that $[\mathfrak{s}, T] = 0$. The number $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r})$ is given by the difference of the dimension of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ and the rank of the matrix $A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r})$, which in this case has the form

$$\begin{pmatrix}
0 & [X_1, X_2] & [X_1, X_3] & [X_1, Z_1] & \cdots & [X_1, Z_r] & 0 & 0 \\
[X_2, X_1] & 0 & [X_2, X_3] & [X_2, Z_1] & \cdots & [X_2, Z_r] & 0 & 0 \\
[X_3, X_1] & [X_3, X_2] & 0 & [X_3, Z_1] & \cdots & [X_3, Z_r] & 0 & 0 \\
[Z_1, X_1] & [Z_1, X_2] & [Z_1, X_3] & 0 & \cdots & 0 & 0 & [T, Z_1] \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
[Z_r, X_1] & [Z_r, X_2] & [Z_r, X_3] & 0 & \cdots & 0 & 0 & [T, Z_r] \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & [T, Y] \\
0 & 0 & 0 & [Z_1, T] & \cdots & [Z_r, T] & [Y, T] & 0
\end{pmatrix} \tag{18}$$

where $\{Z_1, \ldots, Z_r, Y, T\}$ is a basis of $\mathfrak{r}$. Elementary methods show that the determinant of this matrix is the product of $-[T, Y]^2$ and the following determinant

$$\det\begin{pmatrix}
0 & [X_1, X_2] & [X_1, X_3] & [X_1, Z_1] & \cdots & [X_1, Z_r] \\
[X_2, X_1] & 0 & [X_2, X_3] & [X_2, Z_1] & \cdots & [X_2, Z_r] \\
[X_3, X_1] & [X_3, X_2] & 0 & [X_3, Z_1] & \cdots & [X_3, Z_r] \\
[Z_1, X_1] & [Z_1, X_2] & [Z_1, X_3] & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
[Z_r, X_1] & [Z_r, X_2] & [Z_r, X_3] & 0 & \cdots & 0
\end{pmatrix} \tag{19}$$

which must be zero, since the rank of the matrix in (19) gives the number of invariants of the subalgebra $\mathfrak{s}\overrightarrow{\oplus}_{R-2D_0}(rL_1)$, which is non-maximal in virtue of theorem 1. Therefore, the rank of $A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r})$ is less than its dimension, from which the existence of non-trivial invariants is ensured. □

It should be remarked that if R contains only one copy of D_0 or the codimension of $[\mathfrak{r},\mathfrak{r}]$ is $\mathfrak{r}$ which is greater than one, then the conclusion is false, as can easily be extracted from table 1. We will finally see that radicals as considered in this section are only valid in low dimensions in order to obtain Lie algebras $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ such that $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 0$.

Proposition 4. *If* $\dim(\mathfrak{r}) \geqslant 7$ *then* $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) \neq 0$.

Proof. As before, since $[\mathfrak{r},\mathfrak{r}]$ is a codimension one $\mathfrak{s}$-submodule of $\mathfrak{r}$, the action of $\mathfrak{s}$ on a generator $T \in \mathfrak{r} - [\mathfrak{r},\mathfrak{r}]$ is zero. If $\dim(\mathfrak{r}) = 7$, then $\dim(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 10$ and the matrix $A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r})$ has the form

$$\begin{pmatrix} 0 & [X_1,X_2] & [X_1,X_3] & [X_1,Z_1] & \cdots & [X_1,Z_6] & 0 \\ [X_2,X_1] & 0 & [X_2,X_3] & [X_2,Z_1] & \cdots & [X_2,Z_6] & 0 \\ [X_3,X_1] & [X_3,X_2] & 0 & [X_3,Z_1] & \cdots & [X_3,Z_6] & 0 \\ [Z_1,X_1] & [Z_1,X_2] & [Z_1,X_3] & 0 & \cdots & 0 & [T,Z_1] \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ [Z_6,X_1] & [Z_6,X_2] & [Z_6,X_3] & 0 & \cdots & 0 & [T,Z_6] \\ 0 & 0 & 0 & [Z_1,T] & \cdots & [Z_6,T] & 0 \end{pmatrix}. \tag{20}$$

It is easy to verify that the determinant of (20) does not depend on the brackets, and that it is zero. Since for any radical $\mathfrak{r}$ of the considered type such that $\dim(\mathfrak{r}) \geqslant 7$ the determinant of $A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r})$ is a linear combination of matrices of type (20) and matrices as in (19), it follows that $\det A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 0$. □

Observe that this result explains, in terms of the representation theory of $\mathfrak{so}(3)$, why the ten-dimensional Galilei algebra has two (Casimir) invariants depending only on the translations P_i and the pure Galilean transformations K_i.

6. Conclusions

We have seen that for any dimension $n \geqslant 6$, there exist non-semisimple Lie algebras $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ with non-trivial Levi factor $\mathfrak{s}$ such that $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}) = 0$. This constitutes a proof that the Levi decomposition theorem [22] does not reduce the number of generalized Casimir invariants of $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ to some combination of the numbers corresponding to the Levi factor $\mathfrak{s}$ and the radical $\mathfrak{r}$, but depends essentially on the pair $(R,\mathfrak{r})$ formed by the representation R describing the semidirect sum and the radical.

For the rank one simple Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$, the number of invariants of a semidirect sum $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ have been analysed in some detail. In particular, the analysis undertaken has given a representation-theoretic interpretation of the invariants obtained for the (3 + 1) kinematical algebras such as the Galilei algebra. The interest of these Levi factors is therefore justified not only by kinematical problems, but also by the extensions of Bianchi type-IX cosmology [20, 21]. Specially interesting are those admissible extensions which have no invariants. Therefore, invariant quantities for these algebras should be searched using distribution theory [19]. In particular, if the radical is Abelian, we have proved that we will obtain solutions, some of them depending only on variables associated with elements of the radical. This confirms that the fact that the special affine algebras $\mathfrak{sa}(n,\mathbb{R})$ have invariants (for being odd dimensional) is not an isolated case, but also the general pattern for those semidirect sums which are even dimensional. From the computed examples, it seems reasonable to expect that, whenever the radical $\mathfrak{r}$ is a nilpotent Lie algebra, the number of invariants of a semidirect sum $\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ will be nonzero. However, for this case, it is not sufficient to know which is the representation R that describes the semidirect sum. We need more precise information on the structure of $\mathfrak{r}$ (not merely the value of very general invariants such as the nilpotence index), which impedes to establish a general result as for the Abelian case.

The most important question that arises from our results is whether they can be extended to any semisimple Lie algebra of rank $r \geqslant 2$. At least for direct sums of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$, this seems to hold. An example which is worth analysing is the Schrödinger algebra $\mathfrak{S}$ in $(3+1)$ dimensions [28]. Over the basis $\{J_i, K_i, P_i, P_0, C, D\}_{i=1,2,3}$, this algebra is given by the brackets

$$\begin{array}{llll} [J_i, J_j] = \varepsilon_{ijk} J_k & [J_i, K_j] = \varepsilon_{ijk} K_k & [J_i, P_j] = \varepsilon_{ijk} P_k & [K_i, P_0] = P_i \\ [P_i, D] = P_i & [D, K_j] = K_j & [D, P_0] = -2P_0 & [C, P_j] = K_j \\ [C, P_0] = -D & [C, D] = -2C & & \end{array}$$

where P_0 is the time translation, P_i the space translations, J_i the rotations and K_i the pure Galilean transformations. It can easily be verified that the subalgebra $\mathfrak{a}$ generated by $\{K_i, P_i\}_{i=1,2,3}$ is six-dimensional and Abelian, while $\{P_0, C, D\}$ generates a copy of $\mathfrak{sl}(2,\mathbb{R})$. Therefore, we obtain the semisimple algebra $\mathfrak{so}(3) \oplus \mathfrak{sl}(2,\mathbb{R})$, and since $\mathfrak{a}$ is an ideal, we have the Levi decomposition of $\mathfrak{S}$ (by abuse of notation, we can denote the corresponding representation by $D_{\frac{1}{2}} \otimes \mathrm{ad}\,\mathfrak{so}(3)$). If we extract a subsystem of the corresponding system (2), as done in the proof of theorem 2, we obtain that $\mathfrak{S}$ has a fourth-order Casimir operator $\mathcal{P}_4$ depending only on the space translations and pure Galilei transformations

$$\begin{aligned} \mathcal{P}_4 = {} & K_1^2\left(P_2^2 + P_3^2\right) + K_2^2\left(P_1^2 + P_3^2\right) + K_3^2\left(P_1^2 + P_2^2\right) \\ & - 2(P_1P_2K_1K_2 + P_1P_3K_1K_3 + P_2P_3K_2K_3). \end{aligned}$$

For other simple Lie algebras, a direct calculation of the rank of matrices $A(\mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r})$ becomes a enormously difficult problem, and therefore the proofs of the generalization of the results obtained for rank one simple algebras, if they hold, must be approached by completely different means.

Finally, these results are of interest for the study of non-semisimple (maximal) regular subalgebras of simple Lie algebras. The example $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{D_{\frac{1}{2}}\oplus D_0}3L_1$ of table 1 is a regular subalgebra of $\mathfrak{sl}(3,\mathbb{R})$ and has no invariants. It would be important to obtain a detailed description of the non-semisimple maximal regular subalgebras of simple Lie algebras which do not have non-trivial invariants. This problem is of interest not only for symmetry breaking questions [29], but also for solving many fundamental problems which arise in rigidity and contraction theory [5, 30], such as the invariant theory of parabolic subalgebras of semisimple Lie algebras, the construction of contraction trees or the expansion problem [31].

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